5. SAMPLING

5.1. Introduction

It is a well-known fact that today's semiconductor technology is capable to produce digital devices having extremely high complexity at an affordable price, volume and weight. These features can also be utilized for solving tasks related to analog signals if conversion of the analog signal to series of symbols discrete both in time and in amplitude is possible. More precisely, the question is what are the pros and contras of mapping an analog signal into a series of symbols discrete in time and amplitude. In fact, investigations of these problems are covered by the term *sampling*.

Digital devices with computer-like architecture reduce eventually all tasks to operations with binary (two-state) symbols. Any series of symbols (e.g. sampled analog signal or a written text) can be converted into a series of binary symbols. The length of the converted (encoded) series is, however, not indifferent. Possibilities and limits of unambiguous coding are discussed in chapter 6.2.

5.2. The Spectrum of Sampled Series

A widely used method of signal generation is that numbers stored in a computer-like device are periodically converted to electrical quantities, e.g. voltage. Such a device is called digital-to-analog converter (DAC) and it is usually integrated into one circuit. To eliminate unwanted components, the DAC's output signal is smoothed by a filter.

It is an interesting question how the stored numbers have to be chosen to generate signals with specified shape. To specify the task more precisely, let us suppose that a DAC and a smoothing filter are used to generate an absolutely integrable signal with the spectrum $X_e(.)$ by means of so far unknown series of numbers x_i , i = 0, 1, ... entering the DAC input (see Fig. 5.1.). Suppose that the operation of the DAC is periodic in time (*T*). It is also important whether the converter is operated by narrow or by wide impulses.

The latter method is rather practical while the previous one serves as the computational model. Notice that the true digital-to-analog converter can be constructed by an ideal DAC and an ideal smoothing filter with impulse response *m*(.). Using notations of Fig. 5.1., the impulse response of the smoothing filter is

$$h(t)=m(t)*g(t), \qquad t\in(-\infty,\infty)$$

It also follows from the model that there are impulses of the magnitude $x_i T/\Delta$ and width Δ at the output of an ideal DAC. The filters respond to such excitation by the impulse response, so that

$$x(t)=T\sum_i x_i h(t-iT) \tag{5.1}$$

For sake of simplicity, let us assume that the output sample x_i is generated at time *iT*. The output signals of the ideal and the real DAC are also shown in Fig. 5.1.

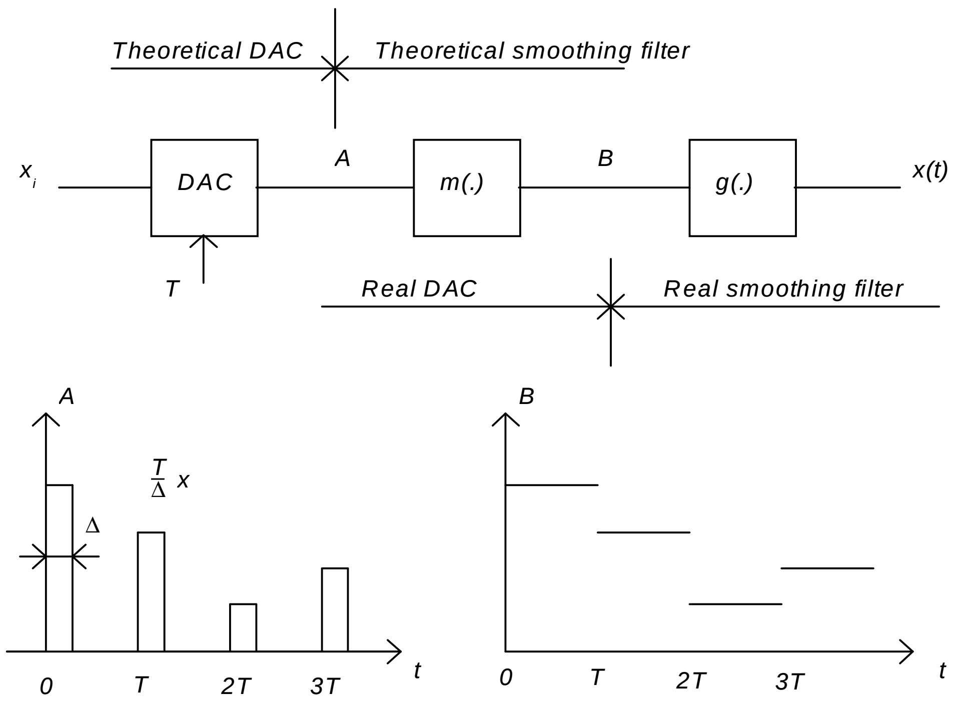

Fig. 5.1. Reconstruction of Analog Signal from Stored Samples

Fourier transform of *x*(.) obviously exists if *h* is absolutely integrable and the sum of x_i (i = 0, 1, ...) exists. In this case

$$x(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2pft} dt = T \sum_i x_i \; H(f)\, e^{-j2pfiT}$$

that is

$$X(f) = H(f) T \sum_i \; x_i\, e^{-j2\pi fiT} \tag{5.2}$$

where *H*(.) is the transfer function of the smoothing filter.

From eq. (5.2) it follows that

$$X_m(f) = T \sum_i \; x_i\, e^{-j2\pi fiT} \tag{5.3}$$

behaves in the same way as if it was the Fourier transform of an absolutely integrable signal. This behaviour establishes the terminology for X_m as being the spectrum of the series x_i, $i = 0, 1, \ldots$

In fact, X_m exhibits the usual symmetry properties of the spectra of the real signals:

$$X_m(-f) = X_m^*(f), \qquad \forall f \in (-\infty,\infty)$$

moreover it is periodic in 1/*T*:

$$X_m(f+\frac{1}{T}) = X_m(f), \quad \forall f \in (-\infty,\infty)$$

Periodicity also means that (5.3) is the Fourier series of $X_m(f)$, i.e. a given X_m can be realized by Fourier decomposition resulting in series of x_i, $i = 0, 1, \ldots$.

So it can be registered that, on the base of stored samples, it is possible to generate a signal with spectrum X_e by means of a DAC and a filter:

$$X_e(f) = H(f)\, X_m(f), \quad \forall f \in (-\infty,\infty)$$

where X_m is a spectrum periodic in 1/*T*.

In the above example , X_e was given and *H* and X_m had to be chosen. In different practical cases the choice depends on the actual situation. The solution is relatively universal if the specification is band limited, e.g.

$$X_e(f) = 0, \quad \text{whenever } |f| > B < 1/(2T).$$

In this case, $H(f)$ may be a lowpass filter (with the cutoff frequency at B) and

$$X_m(f) = \sum_k X_e(f+k/T), \quad \forall f \in (\text{-¥}, \text{¥}).$$

Fig. 5.2. illustrates the relation between X_m and X_e. Furthermore, it shows the passband and stopband of the smoothing filter. It also can be seen that in the case of B < 1/(2T), an unspecified region between the passband and the stopband exists which is needed for the realization of the filter. Samples to be stored can be generated by the Fourier decomposition of the X_m, i = 0, 1, ... :

$$x_i = \int_{-B}^{B} X_m(f)\, e^{j2\pi f i T}\, df = \int_{-\infty}^{\infty} X_e(f)\, e^{j2\pi f i T}\, df \qquad (5.5.)$$

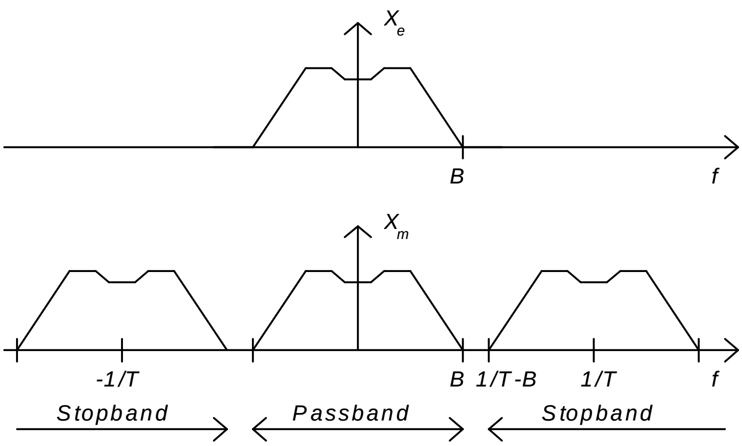

Fig. 5.2. Restoration of the Spectrum X_e

5.3. Signal Reconstruction from Equidistant Samples

Suppose that samples of the signal $x(t)$ were taken so that the time interval between the samples is identical, i.e.

$$x_i = x(iT), \quad i=0, 1, ...$$

In this case the spectrum of the sampled signal (if x is absolutely integrable, then it certainly exists) is:

$$X(f) = T \sum_i x(iT)\, e^{-j2\pi fiT}$$

It may be expected that the spectrum of the sampled series and that of the signal *x* are in a simple relation with each other. Indeed, it is true that

$$X_m(f) = \sum_k X(f+k/T), \quad \forall f \in (-\infty,\infty) \tag{5.6}$$

As the right side of the eq. (5.6) is the function of *f* periodic in 1/*T*, it can be expressed in the form of Fourier series. Computing the coefficients, we obtain exactly $x(i\tau)$, i =0, 1, ... (Q.e.d.).

There is an especially interesting practical case, in which -using some suitable sampling frequency $f_s=1/T$- it is true for all frequencies that only one non-zero element is in eq. (5.6) (the cumulative spectrum is not aliased). That means that the analog signal can be reconstructed from the samples of *x*, equidistant in *T* by means of the system shown in Fig. 5.1., provided that the transfer function matches the spectrum of the signal *x*, i.e.:

$$H(f) = \begin{cases} 1, \text{if } x(f) \neq 0 \\ 0, \text{if} x_m(f) \neq 0 \text{ and } x(f) = 0 \\ \text{arbitrary, otherwise} \end{cases}$$

The straightforward consequence of the above statement is the Shannon's sampling theorem: An absolutely integrable signal can by reconstructed from its samples equidistant in T by means of an ideal lowpass filter with the cutoff frequency *B* < 1/(2*T*).

Shannon's theorem is valid not only for absolutely integrable signals but for harmonic signals and for stationary stochastic processes, too.

5.4. Spectral Density of Random Series

Let us examine the properties of a random signal reconstructed by means of a DAC and a smoothing filter from the series of random numbers. Suppose that ξ_i are probability variables with uniform distribution and zero-mean, moreover that they are uncorrelated, i.e.

$$\mathrm{M}\left(z_i z_j\right) = \begin{cases} s^2, \text{ if } j = i; \\ 0, \text{ if } j \neq i. \end{cases}$$

The latter method is rather practical while the previous one serves as the computational model. Notice that the true digital-to-analog converter can be constructed by an ideal DAC and an ideal smoothing filter with impulse response *m*(.). Using the notations of Fig. 5.1., the impulse response of the smoothing filter is

$$h(t)=m(t)\, g(t), \quad t \in (-\infty,\infty)$$

Suppose the smoothing filter of the signal generator is ideal and its cut-off frequency is 1/(2*T*) so that its frequency response is

$$h(t) = \frac{1}{T}\frac{\sin\left(pt/T\right)}{p\ t/T}, \qquad \forall t \in (-\infty,\infty)$$

Under such a set of conditions it can be proved that the generated signal will be a stationary, zero-mean signal with the spectral density

$$s_\eta(f) = \begin{cases} s^2 T, \text{ if } \left|f\right| < 1/\left(2T\right) \\ 0, \text{ otherwise} \end{cases} \qquad (5.7)$$

5.5. Quantization Noise.

When an analog signal is converted to digital form, each analog sample is replaced by a codeword belonging to a finite set of *N* codewords. This means that there are only *N* different samples which can be exactly represented. It may be a natural (but not necessary) requirement that multiples of a basic unit are assigned to the samples. Generally (but again, not necessarily) *n* bit binary code words are assigned to the samples accordingly to a simple rule. For instance, two-s complement code is such a widely used representation.

Using n bit code words, it is possible to distinguish $N = 2^n$ sample values. The analog-to-digital converter (ADC) generates the following exact sample values:

$$x = \Delta\, i, \quad i = -N/2,...0,1,...N/2-1 \qquad (5.8)$$

Conversion range is an important parameter of such an ADC. If this range is defined by the interval $(-C,C)$ then the magnitude of the quantization steps will be $\Delta = 2C/N$.

Of course, the value of the input sample is usually not equal to any of the discrete values given by (5.8). In fact, the converter substitutes a sample by the codeword representing the value closest to the value of the input sample. Thus the quantized value (x_q) differs from the input value (x): $x_q = x + \varepsilon$, where ε can vary between $-\Delta/2$ and $\Delta/2$.

The difference ε is called the *quantization noise*. In simple models, the quantization noise is modelled by a uniformly distributed probability variable. Furthermore, it is also assumed that the instantaneous values of the quantization noise added to different samples are not correlated, i.e.

$$M(\varepsilon_1 \cdot \varepsilon_2) = 0$$

This model of the quantization noise is useful when the bandwidth of the stationary stochastic signal is relatively great in comparison to the sampling frequency ($B \sim f_s/2$).

In Chapter 5.4. we have seen that the noise process reconstructed from uncorrelated noise samples has a constant spectral density within the range of $|f| < f_s/2$ and its power is equal to that of the samples. Since the signal reconstructing system is linear, the signal appearing at the output is

$$x_q(t) = x(t) + \varepsilon(t)$$

where x(t) is the original input signal. The power of the process $\varepsilon(t)$ can be computed from the distribution of the samples:

$$P_e = M(\varepsilon^2(t)) = M(\varepsilon^2) = \int_{-\Delta/2}^{\Delta/2} x^2 \; f_e(x)dx = \ldots = \frac{\Delta^2}{12}.$$

The subjective measure of the effect caused by the quantization noise can be well defined by the ratio of the reconstructed signal power and the quantization noise power, which is called the *signal-to-noise ratio*. The maximum amplitude of the sinusoidal signal is C so that the maximum power is $P_X = C^2/2$. The signal-to-noise ratio is then

$$\text{SNR} = \frac{P_x}{P_e} = \frac{C^2}{2} : \frac{\Delta^2}{12} = 6(C/\Delta)^2$$

Knowing that $C/\Delta = N/2 = 2^{n-1}$, the signal-to-noise ratio for the maximum amplitude sinewave is

$$\text{SNR} = \frac{3}{2} 2^{2n}, \; i.e. \; \text{SNR}[\text{dB}] = 1.74 + n6.02 \qquad (5.9)$$

Of course, the signal-to-noise ratio is significantly smaller if the power of the converted signal is well bellow the permissible limit.

5.6. Nonlinear Quantization

In a significant part of telecommunication applications, the average power of the sampled and A/D converted signals is within a range of about 36 dB. More precisely, the signal at the ADC input may have the maximum amplitude C but might also have just 1/4000th power (C/64 amplitude, i.e. -36 dB) of the previous one. Should it be required to have 36 dB signal-to-noise ratio even for such a low-level signal, it would result in an unnecessarily great signal-to-noise ratio for high level signals, e.g. for the maximum signal it would be 36 + 36 = 72 dB, which could be satisfied by $n = 12$.

The representation can be made denser if the precise sample values are not chosen as equidistant. In the range of $|x| < C/64 = C_0$, let us have the distance $\Delta_0 = C/32$ so that 64 divisions are in this range. This is just enough to satisfy the 36 dB SNR for the low-level signals. In the next range where $C_0 < |x| < 2C_0$ the distance is doubled to $2\Delta_0$ so that for the signals with the amplitude $2C_0$ the SNR remains the same but the number of samples is only 32 in this range. This procedure can be continued until the entire range $|x| < C$ is covered. It is easy to count that only 256 samples shall be precisely represented using this procedure so that the n = 8 bit code word length meets the above SNR requirement. The price we have to pay for this kind of logarithmic conversion is that the relation between the analog samples and the code words assigned to them is not as easily seen as it was for the linear code. In practice, the logarithmic compression of 8 bit codewords is performed by 13 bit ADCs and an appropriate postprocessing of the obtained samples.

Control questions

1. What conditions are needed to define the spectrum of series of numbers?
2. What are the characteristics of a spectrum of a continuous equidistantly sampled signal?
3. Under what conditions is there no correlation between the samples of quantization noise?

Exercise

1. What is the spectrum of the series $x_i = 2^{-i}$, i =0, 1,... ?

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