2. SIGNALS

2.1. Introduction

Signals are time- and/or place-dependent physical quantities or their mathematical representations which have meaningful content. In this sense, the time-dependent output of an electroacoustical transducer (e.g. a microphone), or the time- and space-dependent sound pressure in a certain point of the field, or even the blackness of a photograph as the function of plane co-ordinates, can be regarded as signal. Functions are obvious mathematical models of the signals. In the simplest but typical case, these functions are scalar (often complex) depending on one variable only (which is usually time). Methods discussed in this chapter refer strictly to these functions although the same methods are used when more general signals (vectorial, multivariable) are described.

2.2. Classification of Signals

Signals frequently used in practice can be classified according to the 'richness' of their domain and range. Back to the classical example, the output signal of a microphone is continuous both in its domain and in its range. Such a signal is called *analog*. Another group of signals exists with instantaneous values making up a finite set of numbers. In this case we speak about discrete range or *discrete amplitude* signals. Another case is when the instantaneous values of the signal are defined only in some time instants (usually at $t = t_0 + kT$, $k = 0, 1, ...$). This signal is said to be *discrete in time*. In fact, this is not a real signal but a series of numbers. In today's telecommunication, signals discrete both in time and amplitude are of huge importance. These signals are called *digital* and their significance lies in the fact that digital signals can be handled by computers.

Another aspect of signal classification may be the purpose the signal is analyzed for. It is a typical task to compare and qualify similarity of two signals, for instance the time-dependent sound pressure and the electrical output signal of a microphone. In such a case, it is obviously impossible to characterize the quality of the microphone by comparing just a single continuous sound, say the vowel 'a', it is necessary to examine several different functions or sets of functions. On top of that, occurrence and significance of the examined functions are not necessarily the same so that signal analysis as the analysis of sets of functions is related also to the terms used in probability calculus. Because theory and terms of stochastic processes give proper frames to such an analysis, the class of *stochastic signals* has been introduced.

On the other hand, if the behaviour of a system can be well judged by the response to a single previously defined function, the analysis is said to be done with a *deterministic* signal. The deterministic signal is a very 'pleasant' analyzing tool if it can be defined by a simple equation. However, the sense of such conclusions is more limited (e.g., perfect transmission of the vowel 'a' does not conclude the same for the vowel 'i'). In short, deterministic signals are concrete functions while a stochastic signal can be interpreted as set of functions, which have some similar characteristic as well.

2.3. Deterministic Signals

To classify and to characterize deterministic signals used in telecommunication practice, some useful categories and terms have to be introduced. Some of these are reviewed in the following.

1.) *Finite time (finite hold)* signal: the $x(t)$, $t\in(-\infty,\infty)$ is said to be finite time if a pair of $t_1>-\infty$, and $t_2<+\infty$, exists, so that $x(t)=0$ for $\forall t<t_1$ and $\forall t>t_2$.

2.) *Absolutely integrable* signal: $x(t)$, $t\in(-\infty,\infty)$ is said to be absolutely integrable if

$$\int_{-\infty}^{+\infty}|x(t)|dt<+\infty.$$

3.) *Energy* signal: the $x(t)$, $t\in(-\infty,\infty)$ is said to have finite energy if

$$E_X=\int_{-\infty}^{+\infty}x^2(t)dt<+\infty$$

4.) *Limited* signal: $x(t)$, $t\in(-\infty,\infty)$ is said to be limited if a $K<+\infty$, exists, so that

$$|x(t)|<K \text{ for } \forall t\in(-\infty,\infty).$$

Note: if there is such time t exists where $|x(t)|=K$ then K is said to be the (absolute) peak value of the signal.

5.) *Finite average* signal: the $x(t)$, $t\in(-\infty,\infty)$ signal is said to have finite average if

$$A_x=\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}x(t)\,dt \text{ exists and is finite.}$$

Note: A_x is called the DC average of the signal.

6.) *Power* signal: the $x(t)$, $t\in(-\infty,\infty)$ signal is said to have a finite average power if

$$P_x=\lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}x^2(t)dt \text{ exists and is finite.}$$

7.) *Periodic* signal: the $x(t)$, $t\in(-\infty,\infty)$ signal is said to be periodic in T if $x(t+T)=x(t)$ for $\forall t$. T is said to be the fundamental period if there no such a $T_o<T$ exists, for which $x(t+T_o)=x(t)$ is valid for $\forall t$.

8.) *Harmonic* (sinusoidal) signal: $x(t)$, $t\in(-\infty,\infty)$ signal is said to be harmonic if for some A, W and F values $x(t)=A\cos(Wt+F)$ for $\forall t$. A is the amplitude, W is the angular frequency and F is the phase of the signal.
Note: In technical literature, harmonic signals are widely expressed in complex form: $x(t)=A\,e^{j(Wt+F)}$.

9.) *Quasi-periodic* signal: $x(t)=\sum_i A_i\cos(w_i t+F_i)$ is said to be quasi-periodic if the ratios of angular frequencies are irrational numbers.

It can be seen from the list above that essentially two groups of deterministic signals are important from the practical point of view: One of them, characterized by features 1.-3. consists of the impulse-like signals (bursts), while the *steady* signals described by features 4.-9. form the other group.

2.4. Spectral Decomposition of Signals

For the analysis of linear time-invariant systems, it is preferable to handle the input (or output) signal as the sum of harmonic signals since in this the case the influence of the system on the signal can easily be estimated. Therefore, it is an important question what are the conditions for a signal to be composed of such harmonic components or of any other composite form. This signal composition is denoted as spectral. Let us recall two characteristic examples of spectral composition.

2.4.1 Fourier-Series of a Periodic Signal

If $x(t)$ is a continuous signal that is periodic in T, it can be expressed as

$$x(t) = \sum_{i} X_i \exp(j2\pi it/T) \tag{2.1}$$

The series of the above function is uniformly convergent and its coefficients are computed as follows

$$X_i = \frac{1}{T}\int_{0}^{T} x(t)\exp(-j2\pi it/T)\,\mathrm{d}t \tag{2.2}$$

2.4.2 Fourier Transform of Absolutely Integrable Signals

Let $x(t)$, $t\in(-\infty,\infty)$ be an absolutely integrable function. In this case, $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, \mathrm{e}^{\mathrm{j}2\pi \mathrm{f}t}\, \mathrm{d}f \tag{2.3}$$

where

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, \mathrm{e}^{-\mathrm{j}2\pi \mathrm{f}t}\, \mathrm{d}t \tag{2.4}$$

The function $X(.)$ is called the Fourier transform of $x(.)$. It can be seen that in this case an integral is used for composition instead of a sum. Although this is a great difference from the mathematical point of view, it is more important that from the technical point of view the two compositions seem to be similar.

Spectral decomposition of signals introduces further practical aspects for signal classification and some specific terms related to the frequency domain:

1. *Band limited* signal: $x(.)$ is said to be band limited within the $f_1 < f_2$ range if it has no components outside the ranges $[f_1, f_2]$ and $[-f_2, -f_1]$.

2. *Narrowband* signal: $x(.)$ is said to be narrowband if $(f_2 - f_1)/f_2 << 1$.

3. *Effective* frequency: that $B > 0$, for which

$$B^2 = \frac{\int_{-\infty}^{+\infty} f^2 |x(f)|^2 df}{\int_{-\infty}^{+\infty} |x(f)|^2 df} \tag{2.5}$$

Notice that the numerator of the eq. (2.5) is the energy of the derived signal and the denominator is the energy of the signal itself. Obviously, the above equation can be extended to the periodic and quasi-periodic signals as well.

2.5. Stochastic Processes

Let us consider the following examples:

Example 1. Suppose the signal appearing on the secondary coil of a line transformer is given by $x_t = A \cos(W t + F)$, where the value of the phase shift Φ Î $[0, 2\pi]$ depends on the moment of switching the power on related to the first negative-to-positive transition of the line voltage time function following the switching. An obvious model can be set up by saying that F is a probability variable uniformly distributed over $[0, 2\pi]$.

Example 2. One possible way of transmission of integer numbers a_i, $i = 0, 1, ...$ is to compose a sum of so called elementary signals $x(.)$ limited to duration T in the following form

$$x_t = \sum_i a_i \, x(t - iT)$$

This signal will differ for various $a = \{a_i, i = 0, 1, ...\}$. The analysis of such a signal on the base of all possible combinations of a may cause difficulties if specific statistical relations between the elements of the a series are not known. It is much more reasonable to make use of such an information and to say (if possible) that the numbers a_i, $i = 0, 1, ...$ are independent, equally distributed probability variables.

The common feature of the above examples is that the signal (a time-dependent function) appeared as an object depending (also) on probability variables. A given probability variable (example 1) or a given series of probability variable (example 2) results in one concrete function. Such a function is called the *realization* of the random signal (a stochastic process) so that the random signal can be considered as a set of concrete functions. It is a certain deficiency of such a view that it hardly reflects what the characteristic and what the specific elements of the set are.

On the other hand, in both examples the observed quantity (the signal) is a probability variable at any moment so that it can be even considered as a set of probability variables with continuous parameter t. It is obvious that for those t-s which are ‘close’ to each other, probability variables with ‘similar’ values belong. (In example 1, too, ‘similar’ values of probability variables belong to those parameters which are about one period of time apart from each other).

Anyway, the above way of thinking is useful even in cases when (unlike in our examples) the signal values cannot be expressed as probability variables. As we shall see, the knowledge of the adequate statistical properties of signal values is enough to answer several important practical questions.

2.5.1. Superficial Characterization of Stochastic Processes

The most important parameter of the random signal ξ are the values of its realization at an arbitrary time t. As this value, x_t, is also a probability variable, its behaviour is characterized by probabilities

$$F_x(x, t) = P(x_t \le x) \tag{2.6}$$

where F_ξ is a two variable function called as one dimensional distribution (or amplitude distribution) of the process x. The range of the F_x is the [0,1] interval and as a function of the first variable it is monotonously increasing and continuous from the right side. If x_t is a probability variable with continuous range (which is typical for analog signals) then

$$f_x(x,t) = \frac{\partial}{\partial_x} F_x(x,t) \tag{2.7}$$

is the so called *one dimensional probability density function* which gives a detailed characterization of the signal behaviour, similar to the distribution function.

Another important question is the probability that an instantaneous signal value gets out of the limits of a given interval, e.g. [-A, A]. Knowing F_x and f_x , the answer is obvious.

The expected value of probability variables x_t gives a superficial but often useful characterization of the signal behaviour:

$$m_\xi = M(x_t) = \int_{-\infty}^{+\infty} x\, f_x(x,t)\, \mathrm{d}x, \tag{2.8}$$

Similarly, an other useful parameter is the expected value of the signal power:

$$M(x_t^2) = \int_{-\infty}^{+\infty} x^2\, f_x(x,t)\, \mathrm{d}x \tag{2.9}$$

The function $m_x(t)$, $t \in (-\infty, \infty)$ is called the time-dependent expected value of the process x.

In certain cases the knowledge of the one-dimensional probability distribution function is not sufficient. For instance, such is the case when knowing ξ_{t_1}, the value of ξ_{t_2} (t_2 ^{1} t_1) has to be determined. Of course, x_{t_2} may be estimated by $m_x(t_2)$ but (especially if t_2 is close to t_1) we obviously do not use reasonable the knowledge of x_{t_2}. The estimation can be more precise if we know the two-dimensional distribution function characterizing the common behaviour of x_{t_1} and x_{t_2}.This function is the vectorial probability distribution

$$F_x(x_1, x_2, t_1, t_2) = P(x_{t_1} < x_1 \text{ and } x_{t_2} < x_2) \tag{2.10}$$

The corresponding two dimensional density function (if it exists) can also be defined:

$$f_x(x_1, x_2, t_1, t_2) = \frac{\partial^2}{\partial_{x_1} \partial_{x_2}} F_x(x_1, x_2, t_1, t_2) \tag{2.11}$$

In some practical cases even the two-dimensional distribution is unknown. The solution of the estimation problem may be obtained, however, if L_x *autocorrelation function* is known

$$L_x(t_1, t_2) = M(x_{t_1}, x_{t_2}), \qquad t_1, t_2 \in (-\infty, \infty) \tag{2.12}$$

It is easy to find such problems where the distribution of the probability variable $x = (x_{t_1}, x_{t_2}, \dots x_{t_n})$ has to be known for solving the problem in full depth. On the other hand, there are also several signals whose probability distribution function completely determined just by two simple functions, namely by the time-dependent expected value and by the autocorrelation function. That is why many important problems can be solved even if not too much is known about them.

2.5.2. Stationary Processes

In the strict sense of the word, processes characterized by distribution functions insensitive to time-shift are called (strongly) ***stationary***. More precisely, the signal is said to be stationary if it is true for all $n > 0$, all series of $t_1, t_2, \dots t_n$, and for all t, that

$$F_\xi^{(n)}(x_1, x_2, .. x_n, t_1+t, t_2+t, .. t_n+t) = F_\xi^{(n)}(x_1, x_2, .. x_n, t_1, t_2, .. t_n)$$

This invariance to time-shift shows the invariability, i.e. homogeneity of the signal in time. Obviously, the parameters are simpler in this case, e.g. the one-dimensional distribution function is reduced to a single variable function

$$F_x^{(1)}(x_1, t_1) = F_x^{(1)}(x_1, 0) = F_x(x)$$

while the two-dimensional distribution is in fact a function of three variables only:

$$F_x^{(2)}(x_1, x_2, t_1, t_2) = F_x^{(2)}(x_1, x_2, 0, t_2-t_1)$$

The expected value is independent of time:

$$M(x_t) = m_x(t) = m_x, \tag{2.13}$$

while the autocorrelation function depends only on the interval between t_1 and t_2:

$$M(x_{t_1}, x_{t_2}) = L(t_1, t_2) = R(t_2 - t_1) \tag{2.14}$$

Moreover, since L_x is a symmetrical function of t_1 and t_2, R_x is an even function.

It may also frequently happen that the distribution functions of a signal are not known but the signal fulfils the conditions given by the equations (2.13) and (2.14). Even in this case it is possible to solve some important practical problems. Processes exhibiting such a character are therefore classified individually and called *weekly stationary* signals.

2.5.3. Ergodic Processes

It is an everyday experience that there are such signals or phenomena, realizations of which are quite different but their character, however, their long-time averages or the characteristic rhythm of their fluctuations are the same or much alike. This feature is so much the more important as there is a chance just for such signals to give a good picture about the behaviour of other realizations by examining *just one* of them. More precisely, a process is called *ergodic* if almost any of its realizations is suitable to deduce any of its distribution functions.

It can be proved that a strongly stationary process is ergodic or can be composed as a mixture of ergodic processes. To estimate the parameters of such processes, time averaging is used. It can be shown for ergodic processes that the average

$$A(x) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} x_t\,dt$$

is equal to the expected value $m(x)$ of the process with a probability equal to one. For non-ergodic processes, this average depends on blind chance, i.e. it depends on constellation of the realization values during the examination period. However, in case of an ergodic process, every value will be realized exactly in the extent corresponding to its probability, i.e. one can expect that the average will result in (almost) the same value for all experiments. For this, it is of course necessary to have a lose coupling among the ‘soldiers of the army’. Ergodicity is ‘pleasant’ feature of the process since the accuracy of the measurement of such a process is limited only by the duration of the measurement.

2.5.4. Linear Transforms of Stationary Processes

Suppose that x is an ergodic process with an expected value m_x. In accordance with the conclusion which was made at the end of the previous chapter, this expected value can be considered a DC-component of the process.

Let us determine the expected value of the signal appearing at a filter output if the input signal of the filter is the process x. The expected value of the output signal is expected to be dependent on m_x and on the DC-gain of the filter.

Suppose $h(.)$ is the impulse response of the filter so that the output signal at time t is

$$h_t = \int_{-\infty}^{+\infty} h(t)\, x_{t-t}\,\mathrm{d}t$$

and its expected value is

$$M(h_t) = M\Big(\int_{-\infty}^{+\infty} h(t)\, x_{t-t}\,\mathrm{d}t\Big)$$

If the impulse response satisfies certain conditions, then the computation of the expected value and the integration can be swapped:

$$M(h_t) = \int_{-\infty}^{+\infty} h(t)\, M(\xi_{t-\tau})\,\mathrm{d}t = \int_{-\infty}^{+\infty} h(t)\, m_x\,\mathrm{d}t$$

As m_x does not depend on the integration, it can be put in front of the sign and the remaining term can be extended by $e^{-j0\tau}$ (=1)

$$m_x(t) = m_x\int_{-\infty}^{+\infty} h(t)\,\mathrm{d}t = m_x\int_{-\infty}^{+\infty} h(t)\,\mathrm{e}^{-j0\tau}\,\mathrm{d}t = m_x H(0)$$

which is exactly the expected result. $H(0)$ is the frequency response of the filter at zero frequency (the Fourier transform of the filter impulse response).

A further interesting question is whether the output autocorrelation function of the filter can be determined provided the autocorrelation function of the stationary input signal is known. Will the signal remain stationary at all? Without losing generality, we restrict to zero-mean processes. Due to the limited space, instead of a detailed discussion, only the result is presented. With appropriate restrictions, the output signal η remains stationary and if the Fourier transform

$$s_x(f) = \int_{-\infty}^{+\infty} R_x(t)\, \mathrm{e}^{-\mathrm{j}2\pi ft}\, \mathrm{d}t \tag{2.15}$$

exists then the Fourier transform of the autocorrelation function of the output signal is

$$s_h(f) = s_x(f)\, |H(f)|^2, \quad \text{if} \quad f \in (-\infty, \infty) \tag{2.16}$$

The Fourier transform of the autocorrelation function is called the *spectral density* function of the stationary process. This is an even function as R_ξ is also even. As it will be shown in the following chapter, the function is also non negative, too.

2.5.5. Physical Meaning of Spectral Density Function

As we have seen previously, deterministic signals can be synthesised by a sum or by an integral of harmonic components. It is an interesting question whether the same could be done with stationary random signals. The answer is positive but results in the definition of a not too expressive new integral term. Instead, we rest contented with an approach which is satisfactory in the majority of practical cases although it is not an alternative to a more thorough survey of spectral decomposition.

Supposing that a stationary process can also be composed of a sum of harmonic signals, it is quite acceptable that the output signal of a narrowband bandpass filter may contain only spectral components falling into the passband of the filter. Let us examine the power of the filtered signal. Using the notation of Fig. 2.1.

$$M(h_t^2) = R(0) = \int_{-\infty}^{+\infty} \mathrm{s}_h(f) e^{-j2p\,f\,0} \mathrm{d}f \;=\; \int_{-\infty}^{+\infty} s_x(f)\, |\, H(f)^2 \,|\, \mathrm{d}f$$

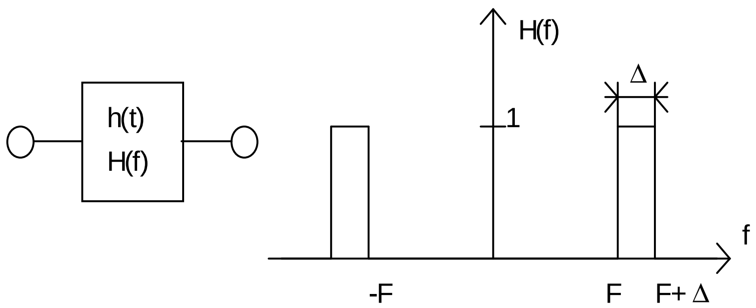

Fig. 2.1 Amplitude Response of an Ideal Bandpass Filter

(Here we exploited that R_h is the inverse Fourier transform of s_h.) Taking also into account that s_x is an even function and using the $H(f)$ shown in the figure

$$M(h_t^2) = 2 \int_{F}^{F+\Delta} s_x(f)\, \mathrm{d}f$$

If the bandwidth Δ is small then according to the theorem of integration calculus

$$M(h_t^2) \cong 2\, s_\xi(F)\, \Delta \qquad (2.17)$$

Since only components having frequency near to F appear at the filter output, the resulting power is given by $M(h_t^2)$. However this quantity depends on the value of $s_x(F)$ representing a kind of spectral intensity distribution.

2.5.6. Relations Between Stochastic Processes

Stochastic process is a parametrical set of probability variables. Relation between two stochastic processes can be described in several ways. Most important features, however, are the mutual independence or dependence of the processes.

Independence: x and h processes are said to be independent, if all the elements of the process x are independent of elements defining the process h.

Consequence: if x and h processes are independent processes then e.g. x_t, i.e. the probability variable denoted by t (the 'soldier' of the 'army' x bearing the 'dog-tag' t) is independent of all the 'honest' (i.e. measurable) functions of probability variables η_t

Dependence: process h is said to be dependent on x if all the processes which can be described by the 'soldiers' of h can be defined solely by the elements of the 'army' x , too. In other words: if any of h_t can be expressed as a function of the probability variables x_t.

Example: Process h is dependent on process x if the relation for any of its probability variable is $h_t = x_t^2$.

$V_t = \int_{-\infty}^{\infty} h(t) \cdot x_{t-t} \cdot dt\,,\ \ t \in (-\infty, \infty)$ is also a process dependent on x, but in this case, every element of ς is a function of *infinite* number of elements of x .

Note: x or x_t , $t \in (-\infty,\infty)$ is a process, while x_t is only a probability variable, a 'soldier' of an 'army'.

2.5.7. Operations on Processes

By processing elements of one or more processes, new processes can be created. An important question is what can be said about the resulting process, what kind of knowledge is necessary to describe some features of the resulting process.

- *Gain and Delay*: Suppose that $\eta_t = A \cdot \xi_{t-T}$, where A is a gain (reciprocal value of attenuation) and T is a delay. If x is at least weakly stationary then h is stationary, too and

$$m_h = A \cdot m_x \quad \text{and} \quad R_h(t) = A^2 \cdot R_x(t)\,.$$

If the spectral power density function of the process x exists then that of the process h exists, too and

$$s_\eta(f) = A^2 \cdot s_\xi(f).$$

Clearly, this is an especially simple linear (invariant) transformation, transfer function of which is

$$H(f) = A \cdot e^{-j2\pi fT}\,.$$

- *Derivation*

Process h, the derivate of the process x is defined by variables

$$h_t = \frac{1}{\Delta}\left(x_t - x_{t-\Delta}\right), \quad if \quad \Delta \to 0 .$$

Time function of the expected value is obviously zero, the autocorrelation function can be derived (assuming stationary input) can be derived:

$$R_h(t) = \frac{1}{\Delta_1 \Delta_2}\left(M(x_t x_{t-t}) - M(x_{t-\Delta_1} x_{t-t}) - M(x_t x_{t-t-\Delta_2}) + M(x_{t-\Delta_1} x_{t-t-\Delta_2})\right)$$

$$R_h(t) = \frac{1}{\Delta_1 \Delta_2}\left(R_x(t) - R_x(t-\Delta_1) - R_x(t+\Delta_2) + R_x(t+\Delta_2-\Delta_1)\right).$$

Since

$$\frac{1}{\Delta_1}\left(R_x(t) - R_x(t-\Delta_1)\right) = R_h^{'}(t) ,$$

one can identify that

$$R_h(t) = \frac{1}{\Delta_2}\left(R_x^{'}(t) - R_x^{'}(t+\Delta_2)\right) \Rightarrow -R_x^{''}(t) .$$

The result corresponds well to the expectation that the spectral power density function is

$$s_\eta(f) = s_\xi(f) \cdot \left|H(f)\right|^2 = s_\xi(f) \cdot \left|j2\pi f\right|^2 ,$$

hence the derivation can be expressed (not easily) in the form of a convolution integral but it can be characterized also by the transfer function, which is just as expected

$$H(f) = j2\pi f .$$

Since not all the autocorrelation functions can be derived (especially not twice), it is not guaranteed that the derivate of an arbitrary stationary process exists at all.

- *Integration*

The operation has to be reformulated in the following way: Let us have a stationary process x and let us try to find a process h, the derivate of which is x.!

Let us examine the process which can be obtained from x by passing it through a filter transfer function of which is

$$H(f) = \frac{1}{j2\pi f}$$

If this process exists and it is stationary then -according to the previous idea- its derivate is really the process x . Proof of existence and of stationarity is not trivial. However, it is obvious that the integral does not exist if

$$\frac{s_\xi(f)}{\left|j2\pi f\right|^2}$$

can not be integrated (i.e. if the process to be integrated contains too intense components in the low frequency region).

- *Sum of Independent Processes*

If $h = x + n$ (where both x and n are stationary) then the expected value of the resulting process is

$$m_\eta = m_\xi + m_\nu,$$

and the autocorrelation function is

$$R_\eta(\tau) = M(\xi_t \xi_{t-\tau}) + M(\xi_t \nu_{t-\tau}) + M(\nu_t \xi_{t-\tau}) + M(\nu_t \nu_{t-\tau}),$$
$$R_\eta(\tau) = R_\xi(\tau) + M(\xi_t)M(\nu_{t-\tau}) + M(\nu_t)M(\xi_{t-\tau}) + R_\nu(\tau),$$

After all

$$R_\eta(\tau) = R_\xi(\tau) + R_\nu(\tau) + 2m_\xi m_\nu.$$

If the spectral power density of the components exists then

$$s_\eta(f) = s_\xi(f) + s_\nu(f).$$

Consequence: power of the sum of independent processes is equal to the sum of the powers of the individual components, in other words: independent processes are added in terms of power (provided the expected value of the components is zero - except one single component).

- *Product of Independent Processes*

$$h = x \cdot n$$

Further, only stationary processes are considered. Because of the independence, the expected value can be computed as

$$m_h = m_x \bullet m_n$$

and the autocorrelation function as

$$R_\eta(\tau) = M(\xi_t \nu_t \cdot \xi_{t-\tau} \nu_{t-\tau}).$$

This product can be regrouped in a more reasonable way:

$$R_\eta(\tau) = M(\xi_t \xi_{t-\tau} \cdot \nu_t \nu_{t-\tau}),$$

Thus (because of the independence)

$$R_\eta(\tau) = R_\xi(\tau) R_\nu(\tau).$$

An example of the application of this result is the double-sided modulated signal

$$\eta = \xi \cdot \cos(2\pi f t + \phi),$$

autocorrelation function of which is

$$R_\eta(\tau) = \frac{1}{2} R_\xi(\tau)\cos(2\pi f\tau).$$

2.5.8. Characterization of Interprocess Relations

It is a frequent practical case that two processes are not independent but they also do not completely determine each other. It is an exciting question then what can be said e.g. about the value of η_t, provided the values of *x* are observed in some time instants or in a broader time interval. If we confine ourselves to linear predictions (functions) based on the values of *x*, then the so called cross correlation function of the two processes can be useful at (or necessary for) their creation:

$$L_{\eta\xi}(t_1, t_2) = M(\xi_{t_1}\eta_{t_2}).$$

It also happens that both *x* and *h* are (at least weakly) stationary, moreover that

$$L_{\eta\xi}(t_1, t_2) = R_{\eta\xi}(t_2 - t_1).$$

In such a case we say that *x* and *h* are stationary alltogether, too.

Example: given is a process

$$h_t = \int_{-\infty}^{\infty} h(t) \cdot x_{t-t} \cdot dt, \quad t \in (-\infty, \infty).$$

and the cross-corelation function of *x* and *h* is to be determined:

$$L_{hx}(t_1, t_2) = M(x_{t_1} \cdot \int_{-\infty}^{\infty} h(t) \cdot x_{t_2 - t} dt) = \int_{-\infty}^{\infty} h(t) \cdot M(x_{t_1} x_{t_2 - t}) dt$$

As it can be seen, the function is dependent only on time difference hence the two processes are stationary alltogether, too and their cross correlation function is
cross correlation function of which is to be determined.

$$L_{hx}(t_1, t_2) = \int_{-\infty}^{\infty} h(t) \cdot R_x(t_2 - t_1 - t) dt \ .$$

As it can be seen, the function is dependent only on time difference hence the two processes are stationary alltogether, too and their cross correlation function is

$$R_{hx}(x) = \int_{-\infty}^{\infty} h(t) \cdot R_x(x - t) dt \ .$$

By the way, this relation gives a possibility for the 'non-destructive' measurement of impulse response of linear systems. (Application of narrow, fast rise impulses in a measurement is sometimes either not desirable or not possible.)

Control Questions

1. When is it reasonable to analyze a system by means of a stochastic signal?
2. How can a stationary signal be characterized?
3. What is the definition of the ergodic process?
4. When is the knowledge of the autocorrelation function essential?
5. What is the definition of the spectral density function?

Exercises

1. A stationary process having the expected value zero and constant spectral density at every frequency $f < B$ is called band-limited white noise. Compute the autocorrelation function of such a process.
2. Compute the spectral density function of the process characterized by the following autocorrelation function: $R_x(t) = R_0 \exp(-|\tau|/T)$

References

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